

Dipolar Bose-Einstein Condensates in Weak Anisotropic Disorder

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Here we study properties of a homogeneous dipolar Bose-Einstein condensate in a weak anisotropic random potential with Lorentzian correlation at zero temperature. To this end we solve perturbatively the Gross-Pitaevskii equation to second order in the random potential strength and obtain analytic results for the disorder ensemble averages of the condensate and the superfluid depletion, the equation of state, and the sound velocity. For a pure contact interaction and a vanishing correlation length, we reproduce the seminal results of Huang and Meng, which were originally derived within a Bogoliubov theory around a disorder-averaged background field. For dipolar interaction and isotropic Lorentzian-correlated disorder, we obtain results which are qualitatively similar to the case of an isotropic Gaussian-correlated disorder. In the case of an anisotropic disorder, the physical observables show characteristic anisotropies which arise from the formation of fragmented dipolar condensates in the local minima of the disorder potential.

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I. INTRODUCTION

Since the realization of Bose-Einstein condensates (BECs) in 1995 [1, 2], there was a significant interest about effects of the disordered potentials on the properties of ultracold quantum gases. The reason for this is not only because of the unavoidable irregularities in the trapping potential induced by wire imperfections [3, 4], but also due to the fact that disorder can be generated and controlled using laser speckles [5, 6]. It is well known that cold atoms are a promising tool for simulating other physical systems [7], in the sense of Feynman's quantum simulator [8]. This applies also to the phenomenon of Anderson localization, which was originally used to microscopically describe the absence of diffusion in terms of disorder [9]. It has a clear BEC analogue [10], which has been directly observed [6, 11]. Also, localization inside BECs due to disorder created by atomic impurities on a lattice was studied theoretically [12] and observed experimentally [13].

For a theoretical analysis of global dirty boson properties, different methods have been used to describe various limits, ranging from the Bogoliubov theory [14–21] to the Parisi replica method [22–26]. It turns out that long-range correlations within both the condensate and the superfluid remain, despite the presence of disorder. However, both quantities are depleted due to the localization of fragmented condensates in the local disorder potential minima. For a strong enough disorder in a homogeneous system, the depletion increases to such an extent that even a critical disorder strength exists, above which a Bose-glass phase appears, consisting only of localized mini-condensates [27–32]. Effects of disorder were also studied for harmonically trapped BECs [29, 31, 33, 34] and BECs in optical lattices [18, 21, 24, 35–37], while the temperature behavior of dirty boson properties was examined in Refs. [14, 17, 22, 26, 32, 33, 36, 38].

Realization of atomic dipolar BECs [39–41] with long-range anisotropic interaction has generated large interest in the theory of dipolar quantum gases [42–50]. Increase in the strength of dipolar interaction is possible by substituting atoms with magnetic dipoles by heteronuclear molecules, which have a strong electric dipolar moment in rovibrational ground state [51], or by inducing radiative coupling by placing dipoles into a resonator [52]. Dipolar condensates were studied in the case of isotropic disorder [30, 53], which yields characteristic anisotropies for both the superfluid density and the sound velocity at zero temperature due to the anisotropy of the dipolar interaction. Typical disorder potentials realized in experiments are anisotropic, and this has been studied, to the best of our knowledge, only numerically and

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for contact interaction [54, 55], whereas a $3d$ isotropic speckle disorder has also recently been proposed [56]. In this paper we analytically study the impact of a weak anisotropic disorder on a polarized dipolar BEC at zero temperature.

To this end we proceed as follows. In Sec. II we calculate the lowest-order corrections of BEC properties due to the presence of disorder within a mean-field approach. For the sake of generality we consider an arbitrary two-particle interaction and a general disorder correlation function. In Sec. III we specialize the developed formalism to dipolar interaction and a Lorentzian-correlated disorder in Fourier space. This yields for both the superfluid density and the sound velocity characteristic anisotropies, which should be measurable in an experiment. In Sec. IV we present our conclusions and outlook for further related research.

II. MEAN-FIELD APPROACH FOR WEAK DISORDER

Bogoliubov quasiparticles and disorder induced fluctuations decouple in the lowest order [14–21] suggesting that disorder corrections can be calculated at zero temperature by neglecting quantum fluctuations and using a mean-field macroscopic wave function $\psi(\mathbf{r})$ governed by the time-independent Gross-Pitaevskii (GP) equation [53]:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) - \mu + \int d^3r' V(\mathbf{r} - \mathbf{r}')\psi^*(\mathbf{r}')\psi(\mathbf{r}')\right)\psi(\mathbf{r}) = 0. \quad (1)$$

Here, m stands for the particle mass, μ denotes the chemical potential, $V(\mathbf{r} - \mathbf{r}')$ represents an arbitrary two-particle potential, while $U(\mathbf{r})$ describes an external disorder potential. Denoting the disorder ensemble average as $\langle \bullet \rangle$, a homogeneous disordered system can be described, without any loss of generality, by a vanishing mean value $\langle U(\mathbf{r}) \rangle = 0$ and an arbitrary correlation function $\langle U(\mathbf{r})U(\mathbf{r}') \rangle = R(\mathbf{r} - \mathbf{r}')$. With this the GP Eq. (1) is a stochastic nonlinear partial differential equation, where the statistics of the condensate wave function $\psi(\mathbf{r})$ is governed by the statistics of the disorder potential $U(\mathbf{r})$ [27]. Since $\psi(\mathbf{r})$ describes the macroscopic occupation of the ground state, we assume it, without loss of generality, to be real. In addition to the statistical properties of the random potential we will assume that the macroscopic value of some physical quantity P_{mac} , obtained by coarse-graining of a microscopic quantity $P(\mathbf{r})$, over a large volume V , gives the same result as the disorder ensemble average, namely:

$$A_{\text{mac}} = \frac{1}{V} \int_V d^3r A(\mathbf{r}) = \langle A \rangle. \quad (2)$$

Here the length of the coarse-graining $l \sim V^{1/3}$ is assumed to be larger than both the correlation length σ of the disorder potential $U(\mathbf{r})$, and the healing length $\xi = \sqrt{\hbar^2/(2mn g)}$, which represents the characteristic distance at which the condensate wave-function responds to some perturbation in the external potential:

$$l \gg \sigma, \xi. \quad (3)$$

In the definition of the healing length n represents the density of the fluid and $g = 4\pi\hbar^2 a_s/m$ denotes the strength of a short-range interaction part of the two-particle interaction potential $V(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}') + \dots$, expressed in terms of the s-wave scattering length a_s .

We consider the case of a sufficiently small random potential $U(\mathbf{r}) \ll gn$, when the perturbative decomposition of the wave function of the system is justified:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \psi_1(\mathbf{r}) + \psi_2(\mathbf{r}) + \dots, \quad (4)$$

where $\psi_n(\mathbf{r})$ corresponds to the correction of the wave function of order n in the disorder. Solving the GP equation (1) in the zeroth order of $U(\mathbf{r})$ gives

$$\psi_0^2 = \frac{\mu}{V(\mathbf{k}=0)}, \quad (5)$$

whereas the first order correction is straight-forwardly calculated and its Fourier transform reads

$$\psi_1(\mathbf{k}) = -\frac{\psi_0 U(\mathbf{k})}{\frac{\hbar^2 k^2}{2m} + 2\psi_0^2 V(\mathbf{k})}. \quad (6)$$

We note that its disorder ensemble average vanishes. Therefore, we also have to determine the second-order result, which turns out to be

$$\psi_2(\mathbf{k}) = -\int \frac{d^3k'}{(2\pi)^3} \frac{U(\mathbf{k} - \mathbf{k}')\psi_1(\mathbf{k}') + \psi_0[2V(\mathbf{k}') + V(\mathbf{k})]\psi_1(\mathbf{k}')\psi_1(\mathbf{k} - \mathbf{k}')}{\frac{\hbar^2 k^2}{2m} + 2\psi_0^2 V(\mathbf{k})}. \quad (7)$$

In the following we use this systematic perturbative approach and calculate the respective properties of the dirty BEC.

A. Order parameter and condensate depletion

In analogy to quantum field theory, the one-particle density matrix is defined as $\langle \psi(\mathbf{r})\psi(\mathbf{r}') \rangle$ [32]. The macroscopic fluid density is the diagonal part of the one-particle density matrix, $n = \langle \psi^2(\mathbf{r}) \rangle$, whereas the condensate density is usually defined as the off-diagonal long-range order (ODLRO) parameter [32]

$$n_0 = \lim_{|\mathbf{r}-\mathbf{r}'| \rightarrow \infty} \langle \psi(\mathbf{r})\psi(\mathbf{r}') \rangle. \quad (8)$$

Coarse-graining of the one-particle density matrix $\langle \psi(\mathbf{r})\psi(\mathbf{r}') \rangle$ over the fixed volume V before taking the limit does not change the result

$$n_0 = \lim_{|\mathbf{r}-\mathbf{r}'| \rightarrow \infty} \frac{1}{V^2} \int_{V \otimes V} d^3\mathbf{r}_1 d^3\mathbf{r}_2 \langle \psi(\mathbf{r} + \mathbf{r}_1)\psi(\mathbf{r}' + \mathbf{r}_2) \rangle. \quad (9)$$

The integration commutes with the disorder ensemble average, and using Eq. (2), we obtain

$$n_0 = \langle \langle \psi(\mathbf{r}) \rangle \langle \psi(\mathbf{r}') \rangle \rangle = \langle \psi(\mathbf{r}) \rangle^2. \quad (10)$$

The last equality follows from the fact that the average of an already averaged expression can be omitted. Therefore, the depletion of the condensate due to disorder, $n - n_0 = \langle \psi^2 \rangle - \langle \psi \rangle^2$, is simply the variance of the wavefunction. Physically, this condensate depletion is due to the formation of fragmented condensates in the respective local minima of the random potential. Defining a separate Bose-glass order parameter by considering the ODLRO parameter of the two-particle density matrix [32]

$$(n_0 + q)^2 = \lim_{|\mathbf{r}-\mathbf{r}'| \rightarrow \infty} \langle \psi(\mathbf{r})^2 \psi(\mathbf{r}')^2 \rangle = n^2, \quad (11)$$

shows that the density of the fragmented condensates q defined in Eq. (11) coincides with the condensate depletion $n - n_0$. To this end the disorder ensemble average is obtained along the same lines as Eqs. (8)–(10). Thus, we conclude that the localization phenomenon for quenched disorder follows already from a mean-field description of the dirty boson problem. Therefore, our mean-field approach represents a simplified derivation of the disorder-induced condensate depletion in comparison with the Bogoliubov theory of Refs. [14–21]. Note that disorder effects on Bogoliubov quasiparticles have recently been analyzed in Refs. [19, 21].

The perturbative expansion (4) now yields for the particle density

$$n = \langle \psi(\mathbf{r})^2 \rangle = \psi_0^2 + \langle \psi_1(\mathbf{r})^2 \rangle + 2\psi_0 \langle \psi_2(\mathbf{r}) \rangle + \dots, \quad (12)$$

and, correspondingly, for the condensate density

$$n_0 = \langle \psi(\mathbf{r}) \rangle^2 = \psi_0^2 + 2\psi_0 \langle \psi_2(\mathbf{r}) \rangle + \dots. \quad (13)$$

This yields the condensate depletion

$$n - n_0 = \langle \psi_1(\mathbf{r})^2 \rangle + \dots. \quad (14)$$

Using Eq. (6) we arrive to the following expression:

$$n - n_0 = n \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left[\frac{\hbar^2 k^2}{2m} + 2nV(\mathbf{k})\right]^2} + \dots. \quad (15)$$

Note that this represents a result for the condensate depletion in second order of the disorder potential for an arbitrary two-particle interaction potential and an arbitrary disorder correlation function. Specializing to the delta-correlated disorder $R(\mathbf{k}) = R$, and contact interaction $V(\mathbf{k}) = g$, Eq. (15) reduces to

$$n - n_0 = n_{\text{HM}} = \frac{m^{\frac{3}{2}} R \sqrt{n}}{4\pi \hbar^3 \sqrt{g}}, \quad (16)$$

which is the seminal result originally obtained by Huang and Meng within the Bogoliubov theory of dirty bosons [14].

B. Equation of state

Solving the equation $\langle \psi^2(\mu_b) \rangle = n(\mu_b)$ for the chemical potential μ_b yields its dependence on the average fluid density $\mu_b = \mu_b(n)$. We have introduced the notation μ_b , denoting the "bare" chemical potential, because it diverges for uncorrelated disorder, regardless of the density n , as can be seen from inserting expressions (5)–(7) into the second-order correction (12):

$$\mu_b = nV(\mathbf{k}=0) - \int \frac{d^3k}{(2\pi)^3} \frac{\frac{\hbar^2 k^2}{2m} R(\mathbf{k})}{\left[\frac{\hbar^2 k^2}{2m} + 2nV(\mathbf{k})\right]^2} + \dots \quad (17)$$

This unphysical ultraviolet divergence can be removed by renormalising the chemical potential [17]. If the density of the system vanishes, i.e. if there are no particles in the system, the energy needed for a particle to be added also has to vanish $\mu(n=0) = 0$. Therefore, we define the renormalized chemical potential according to

$$\mu(n) = \mu_b(n) - \mu_b(0). \quad (18)$$

With this we obtain, in second order of the disorder strength, the renormalized chemical potential:

$$\mu = nV(\mathbf{k}=0) + 4n \int \frac{d^3k}{(2\pi)^3} \frac{V(\mathbf{k})R(\mathbf{k}) \left(\frac{\hbar^2 k^2}{2m} + nV(\mathbf{k})\right)}{\left[\frac{\hbar^2 k^2}{2m} + 2nV(\mathbf{k})\right]^2} + \dots, \quad (19)$$

which does not contain an ultraviolet divergence.

For calculating the sound velocity later on we will also need the expression for the compressibility of the fluid, or its inverse given by $\partial\mu/\partial n$. Note that from Eq. (18) it follows that the obtained result does not depend on whether we use μ or μ_b . Thus, from the perturbative expansion (19) we read off:

$$\frac{\partial\mu}{\partial n} = V(\mathbf{k}=0) + 4 \int \frac{d^3k}{(2\pi)^3} \frac{\frac{\hbar^2 k^2}{2m} R(\mathbf{k})V(\mathbf{k})}{\left[\frac{\hbar^2 k^2}{2m} + 2nV(\mathbf{k})\right]^3} + \dots \quad (20)$$

C. Superfluidity

Without disorder and at $T = 0$, the whole system is in a superfluid state, moving with an arbitrary wavevector \mathbf{k}_S , which corresponds to the superfluid velocity $\mathbf{v}_S = \hbar\mathbf{k}_S/m$. By introducing disorder that moves with the velocity $\hbar\mathbf{k}_U/m$, some part of the fluid will be moving together with it. The normal, i.e. non-superfluid, component of the fluid n_N is defined as the part that moves together with the disorder, while the superfluid component n_S is defined as the fraction of the fluid that moves with the superfluid wavevector \mathbf{k}_S . Therefore, the macroscopic current density $\langle \mathbf{j}(\mathbf{r}) \rangle$ can be separated in this two-fluid picture as follows:

$$\langle \mathbf{j}(\mathbf{r}) \rangle = n_S \mathbf{k}_S + n_N \mathbf{k}_U. \quad (21)$$

The averaged current density $\langle \mathbf{j}(\mathbf{r}) \rangle$ can be obtained by analyzing the underlying time-dependent GP equation for the system:

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + U\left(\mathbf{r} - \mathbf{k}_U \frac{\hbar}{m} t\right) + \int d^3r' V(\mathbf{r} - \mathbf{r}') \Psi_S^*(\mathbf{r}', t) \Psi_S(\mathbf{r}', t) \right] \Psi_S(\mathbf{r}, t) = i\hbar \frac{\partial \Psi_S(\mathbf{r}, t)}{\partial t}, \quad (22)$$

where the condensate wave function Ψ_S is a product of some as yet unknown function ψ_S and a plane wave with wavevector \mathbf{k}_S that corresponds to the clean-case solution:

$$\Psi_S(\mathbf{r}, t) = e^{i\mathbf{k}_S \mathbf{r}} \psi_S(\mathbf{r}, t) e^{-\frac{i}{\hbar} \left(\mu + \frac{\hbar^2 \mathbf{k}_S^2}{2m} \right) t}. \quad (23)$$

Substituting the ansatz (23) into Eq. (22), changing variables via $\mathbf{x} = \mathbf{r} - \mathbf{k}_U \frac{\hbar}{m} t$ and introducing $\mathbf{K} = \mathbf{k}_S - \mathbf{k}_U$ leads to

$$\left[-\frac{\hbar^2}{2m} \nabla^2 - i \frac{\hbar^2}{m} \mathbf{K} \nabla + U(\mathbf{x}) - \mu + \int d^3x' V(\mathbf{x} - \mathbf{x}') \psi_S^*(\mathbf{x}') \psi_S(\mathbf{x}') \right] \psi_S(\mathbf{x}) = 0. \quad (24)$$

Although ψ_S should in general depend on t , it can be shown via mathematical induction on the perturbative solution that all orders of $\psi_S(\mathbf{x}, t)$ turn out to be time-independent [57]. Note that ψ_S does not depend explicitly on the wavevectors \mathbf{k}_S and \mathbf{k}_U , but only on their difference \mathbf{K} . Here, we are only interested in small values of \mathbf{K} and, therefore, expand $\psi_S = \psi + \mathbf{p}\mathbf{K} + \dots$, with $\mathbf{p} = (\partial\psi_S/\partial\mathbf{K})_{\mathbf{K}=0}$. An explicit equation for \mathbf{p} can be obtained by performing the derivative of Eq. (24) with respect to \mathbf{K} , yielding

$$-\frac{\hbar^2}{2m}\nabla^2\mathbf{p}(\mathbf{x}) - \frac{i\hbar^2}{m}\nabla\psi(\mathbf{x}) + [U(\mathbf{x}) - \mu]\mathbf{p}(\mathbf{x}) + \int d^3x' V(\mathbf{x} - \mathbf{x}') \{[\mathbf{p}^*(\mathbf{x}') + \mathbf{p}(\mathbf{x}')] \psi(\mathbf{x}')\psi(\mathbf{x}) + \psi(\mathbf{x}')^2\mathbf{p}(\mathbf{x})\} = 0. \quad (25)$$

If we take into account Eq. (23), the standard definition of the current density

$$\langle \mathbf{j} \rangle = \frac{1}{2i} \langle \Psi_S^* \nabla \Psi_S - \Psi_S \nabla \Psi_S^* \rangle \quad (26)$$

transforms into

$$\langle \mathbf{j} \rangle = \langle \psi_S^* \psi_S \rangle \mathbf{k}_S + \frac{1}{2i} \langle \psi_S^* \nabla \psi_S - \psi_S \nabla \psi_S^* \rangle, \quad (27)$$

which then can be further reduced to

$$\langle \mathbf{j} \rangle = n\mathbf{k}_S + (\langle \psi \nabla \otimes \text{Im } \mathbf{p} \rangle - \langle \nabla \psi \otimes \text{Im } \mathbf{p} \rangle) \mathbf{K} + \dots \quad (28)$$

In the last line we have neglected higher than linear orders in \mathbf{k}_U and \mathbf{k}_S .

For small disorder strengths, if we expand Eq. (28) with respect to U up to second order, take into account the homogeneity of our problem that leads to $\partial_i \langle \mathbf{p}_2(\mathbf{x}) \rangle = 0$ and note that in zeroth order ψ_S does not depend on \mathbf{K} , thus leading to $\mathbf{p}_0 = 0$, we obtain:

$$\langle \mathbf{j} \rangle = n\mathbf{k}_S + (\langle \psi_1 \nabla \otimes \text{Im } \mathbf{p}_1 \rangle - \langle \nabla \psi_1 \otimes \text{Im } \mathbf{p}_1 \rangle) \mathbf{K} + \dots, \quad (29)$$

where also \mathbf{p} is expanded in the disorder strength as $\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_1 + \mathbf{p}_2 + \dots$. Solving the imaginary part of Eq. (25) in the first order in U gives the Fourier transform of the $\text{Im } \mathbf{p}_1$:

$$(\text{Im } \mathbf{p}_1)(\mathbf{k}) = 2i \frac{\mathbf{k}}{k^2} \psi_1(\mathbf{k}). \quad (30)$$

Thus, together with the solution for ψ_1 given by Eq. (6), we obtain from Eq. (29) and comparison with Eq. (21) the normal fluid density in the form

$$\hat{n}_N = 4n \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k} \otimes \mathbf{k}}{k^2} \frac{R(\mathbf{k})}{\left[\frac{\hbar^2 k^2}{2m} + 2nV(\mathbf{k})\right]^2} + \dots \quad (31)$$

Note that in general the non-superfluid component is represented by a tensor [58].

In the case of a cylindrically symmetric system, we can choose the symmetry axis as the z -axis and denote the polar and the azimuth angle by θ and φ , so integrating Eq. (31) in spherical coordinates with respect to φ yields the angle dependence

$$\begin{aligned} \sin \theta \int_0^{2\pi} d\varphi \mathbf{e}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}^T &= \sin \theta \int_0^{2\pi} d\varphi \begin{pmatrix} \sin^2 \theta \cos^2 \varphi & \sin^2 \theta \sin \varphi \cos \varphi & \sin \theta \cos \theta \cos \varphi \\ \sin^2 \theta \sin \varphi \cos \varphi & \sin^2 \theta \sin^2 \varphi & \sin \theta \cos \theta \sin \varphi \\ \sin \theta \cos \theta \cos \varphi & \sin \theta \cos \theta \sin \varphi & \cos^2 \theta \end{pmatrix} \\ &= \sin \theta \begin{pmatrix} \pi(1 - \cos^2 \theta) & 0 & 0 \\ 0 & \pi(1 - \cos^2 \theta) & 0 \\ 0 & 0 & 2\pi \cos^2 \theta \end{pmatrix}. \end{aligned} \quad (32)$$

If both $V(\mathbf{k})$ and $R(\mathbf{k})$ are θ -independent, i.e. if we have spherical symmetry, integrating Eq. (32) with respect to θ leads to a solution in second order of the disorder potential [14–21]:

$$\hat{n}_N = \frac{4}{3}(n - n_0)\hat{I}. \quad (33)$$

This result shows that the superfluid depletion will be by a factor of 4/3 larger than the condensate depletion. Thus, the localized fragmented part of the fluid hinders the superfluid to move.

D. Sound velocity

In the mean-field approach, we can also define the sound velocity by perturbing the time-independent solution with a small time-dependent variation. It is expected that sound waves with wavelengths of the order of the correlation length would scatter and interfere due to disorder hills and valleys, making the sound velocity impossible to define precisely. Locally, the sound waves would have the same speed as in the clean case. For sound waves with wavelengths much larger than the disorder correlation length, the sound velocity can be calculated using the hydrodynamical approach [53]. Hydrodynamic equations are valid in the macroscopic regime and can be used only for slowly varying quantities that do not depend on the specific microscopic realization. Spatial averaging over distances much larger than the correlation length and much smaller than the wavelength solves the problem. Assuming that it gives the same result as the disorder ensemble average, we obtain the hydrodynamic equations for the macroscopic, i.e. disorder averaged, quantities in the form

$$\frac{\partial n(\mathbf{x}, t)}{\partial t} + \nabla(\hat{n}_S(\mathbf{x}, t)\mathbf{v}_S(\mathbf{x}, t)) = 0, \quad (34)$$

$$m\frac{\partial \mathbf{v}_S(\mathbf{x}, t)}{\partial t} + \nabla\left(\frac{m\mathbf{v}_S(\mathbf{x}, t)^2}{2} + \mu(n(\mathbf{x}, t))\right) = 0, \quad (35)$$

where n denotes the macroscopic density, and the disorder velocity \mathbf{k}_U is taken to be zero. If we write densities and the superfluid velocity as sums of homogeneous equilibrium values and small variations,

$$n(\mathbf{x}, t) = n + \delta n(\mathbf{x}, t), \quad (36)$$

$$\hat{n}_S(\mathbf{x}, t) = \hat{n}_S + \delta \hat{n}_S(\mathbf{x}, t), \quad (37)$$

$$\mathbf{v}_S(\mathbf{x}, t) = \delta \mathbf{v}_S(\mathbf{x}, t), \quad (38)$$

as well as neglect second-order terms in the variations, we get the following linearized system of equations:

$$\frac{\partial \delta n(\mathbf{x}, t)}{\partial t} + \nabla(\hat{n}_S \delta \mathbf{v}_S(\mathbf{x}, t)) = 0, \quad (39)$$

$$\frac{\partial \delta \mathbf{v}_S(\mathbf{x}, t)}{\partial t} = -\frac{1}{m} \nabla \mu(n + \delta n(\mathbf{x}, t)) = -\frac{1}{m} \frac{\partial \mu}{\partial n} \nabla \delta n(\mathbf{x}, t). \quad (40)$$

Taking the time derivative of Eq. (39) and substituting the expression for the superfluid velocity variation from Eq. (40) we obtain the generalized wave equation

$$\frac{\partial^2 \delta n(\mathbf{x}, t)}{\partial t^2} - \frac{1}{m} \frac{\partial \mu}{\partial n} \nabla(\hat{n}_S \nabla \delta n(\mathbf{x}, t)) = 0. \quad (41)$$

From the above equation we deduce that the sound velocity in the direction of on unit vector \mathbf{q} is given by

$$c_{\mathbf{q}}^2 = \frac{1}{m} \frac{\partial \mu}{\partial n} \mathbf{q}^T \hat{n}_S \mathbf{q}, \quad (42)$$

where the tensorial property of the superfluid density has been taken into account. In order to further evaluate the sound velocity (42) for small disorder, the perturbative results for both the inverse compressibility (20) and the superfluid density following from (31) have to be taken into account.

III. DIPOLAR INTERACTION AND LORENTZ-CORRELATED DISORDER

In this section we will specialize the previously developed perturbative formalism and consider BEC systems in the presence of two different anisotropies, namely an anisotropic dipolar interaction between the analyzed particles and an anisotropic disorder potential. The latter is widely studied and physically motivated, for instance, by the anisotropy of the laser-speckle potential [5, 6]. In order to obtain analytical results, we model the disorder correlation function by a cylindrically-symmetric Lorentzian in a Fourier space

$$R(\mathbf{k}) = \frac{R}{1 + \sigma_\rho^2 k_\rho^2 + \sigma_z^2 k_z^2}. \quad (43)$$

The lengths σ_ρ and σ_z are respectively the perpendicular and the parallel correlation length. This function is not physically realistic, but the corresponding results qualitatively coincide with the case of an isotropic Gaussian-correlated

disorder, which was numerically calculated in Ref. [53], and we expect that all phenomena that appear here would also appear qualitatively for a true laser-speckle correlation function in a setup where it decays monotonously with distance.

Assuming that the van der Waals forces between the atoms can be approximated at low energies by an effective contact interaction, the interaction potential in the presence of an external field that aligns the dipoles in a direction \mathbf{m} takes the form [59]

$$V(\mathbf{r}) = g\delta(\mathbf{r}) + \frac{C_{dd}}{4\pi r^3} (1 - 3\cos\phi(\mathbf{m}, \mathbf{r})) , \quad (44)$$

where $\phi(\mathbf{m}, \mathbf{r})$ represents the angle between vectors \mathbf{m} and \mathbf{r} , and C_{dd} denotes the dipole-dipole interaction strength. In the case of magnetic dipoles $C_{dd} = \mu_0 m^2$, with μ_0 being the magnetic permeability and the magnetic dipole moment m , whereas for electric dipoles we have $C_{dd} = d^2/\epsilon_0$, with the vacuum permeability ϵ_0 and the electric dipole moment d . Introducing the ratio of the dipole-dipole and the contact interaction $\epsilon = C_{dd}/3g$, and taking the Fourier transform of the potential, we obtain [42]

$$V(\mathbf{k}) = g \{ 1 + \epsilon [3\cos^2\phi(\mathbf{m}, \mathbf{k}) - 1] \} . \quad (45)$$

The Huang and Meng result [14] for the condensate depletion (16) is linear in R , and therefore we will compare the relative change of physical quantities due to disorder to the relative change of the condensate density. To this end we define a dimensionless disorder correction of the quantity A as

$$\Delta_A = \lim_{R \rightarrow 0} \frac{\frac{A}{A_0} - A_d}{\frac{n_{HM}}{n}} . \quad (46)$$

Here A_0 stands for the quantity A in the clean, isotropic system, while A_d denotes a possible dimensionless anisotropy factor due to the dipolar interaction and n_{HM} represents the Huang-Meng depletion from Eq. (16). Corrections defined in this way are expressed in terms of only three parameters: the relative dipole-dipole interaction strength ϵ , and the correlation lengths in units of the healing lengths, $z_{\rho,z} = \sqrt{2}\sigma_{\rho,z}/\xi$. We consider systems with an overall cylindrical symmetry, where the disorder symmetry axis is parallel to the direction of dipoles. Otherwise the angle between them would be a fourth parameter that would have to be taken into account.

The general case can be calculated analytically for all observables of interest, but the results are too cumbersome to be displayed here [57]. In the following we restrict the results and their discussion to two special cases, namely the pure contact interaction with anisotropic disorder, and the dipolar interaction with isotropic disorder. The general situation of having both dipolar interaction and anisotropic disorder just leads to quantitative but not qualitative changes.

A. Condensate depletion

We now use the disorder correlation function and the interaction potential defined by Eqs. (43) and (45) in order to calculate the disorder corrections of the condensate density, chemical potential, inverse compressibility, and superfluid density, given by Eqs. (15), (19), (20), and (31). Taking into account Eqs. (16) and (46), making a substitution $\mathbf{k} \rightarrow \mathbf{k}\xi/\sqrt{2}$, denoting the direction of the disorder symmetry by \mathbf{d} and introducing direction-dependent anisotropy functions r and v

$$r = \sqrt{z_\rho^2 \sin^2\phi(\mathbf{d}, \mathbf{k}) + z_z^2 \cos^2\phi(\mathbf{d}, \mathbf{k})} , \quad (47)$$

$$v = \sqrt{1 + \epsilon (3\cos^2\phi(\mathbf{m}, \mathbf{k}) - 1)} , \quad (48)$$

Eq. (15) yields the dimensionless disorder correction for the condensate depletion in the second order

$$-\Delta_{n_0} = 8\pi \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\mathbf{k}^2 + v^2)^2 (1 + \mathbf{k}^2 r^2)} . \quad (49)$$

Assuming that the direction of dipoles is parallel to the disorder symmetry, i.e. $\mathbf{d} \parallel \mathbf{m}$, yields that the whole system is cylindrically symmetric. Writing Eq. (49) in spherical coordinates (k, θ, φ) , integrating with respect to k and φ and changing the variable $t = \cos\theta$, leads to:

$$-\Delta_{n_0} = \int_0^1 dt \frac{1}{v(1 + vr)^2} \quad (50)$$

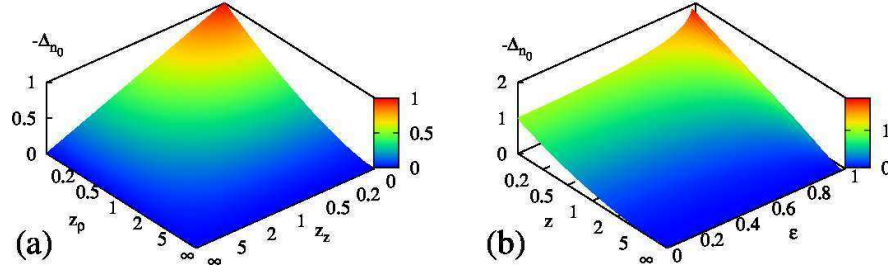


FIG. 1: (Color online) Condensate depletion due to weak disorder for: (a) anisotropic disorder and contact interaction from Eq. (53) and (b) isotropic disorder and dipolar interaction from Eq. (54).

with functions r and v from Eqs. (47) and (48) having the new form

$$r = \sqrt{z_\rho^2 + (z_z^2 - z_\rho^2)t^2}, \quad (51)$$

$$v = \sqrt{1 - \epsilon + 3\epsilon t^2}. \quad (52)$$

The two special cases of contact interaction ($v = 1$) and isotropic disorder ($r = z$) can be solved by using Euler substitutions $r = xt + z_\rho$ and $v = xt + \sqrt{1 - \epsilon}$, respectively, in Eq. (50), which leads to an integral of a rational function with respect to x . Analytic results in the two cases are given by:

$$-\Delta_{n_0}(\epsilon = 0) = \frac{1}{(z_\rho - 1)(z_\rho + 1)} \left[\frac{2z_\rho^2}{(z_\rho + 1)(z_\rho + z_z)} T\left(\frac{z_\rho - 1}{z_\rho + 1} \frac{z_z - z_\rho}{z_z + z_\rho}\right) - \frac{1}{z_z + 1} \right], \quad (53)$$

$$-\Delta_{n_0}(z_\rho = z_z = z) = \frac{z(1 - \lambda)}{(-1 + z^2\delta^2)(1 - \lambda + z\delta(1 + \lambda))} + \frac{(-1 + \lambda)}{\delta(-1 + z\delta)(1 + z\delta)^2} T\left(\frac{z\delta - 1}{z\delta + 1} \lambda\right), \quad (54)$$

with δ and λ introduced by $\delta^2 = 1 - \epsilon$ and $\epsilon = \frac{4\lambda}{1 - 2\lambda + 3\lambda^2}$, to simplify the expressions, and $T(x) = \frac{\arctan \sqrt{x}}{\sqrt{x}}$ being a new function well defined for positive values and analytically continuable for $-1 < x < 0$.

In Fig. 1 we have displayed the results for the disorder correction of the condensate depletion (53) and (54). Note that for $\epsilon = 0$ and $z = 0$ the result is 1, meaning that it coincides with the Huang and Meng result. For increasing correlation length the depletion decreases, and vanishes for infinite correlation length, as expected, since then the disorder is flat. The tiny asymmetry in Fig. 1 a) comes from the fact that z_ρ describes two spatial dimensions and, therefore, has a more pronounced effect on the depletion than z_z which represents only one spatial dimension. Increasing the relative dipole-dipole interaction leads to a larger depletion and, eventually, when it reaches the same order as the contact interaction, the BEC collapses, which is seen in our model as a divergence in Fig. 1 b). Note that the condensate depletion can be measured in matter interference experiments, where the fragmented part of the fluid contributes with a random phase and, therefore, reduces the contrast of the interference pattern.

B. Equation of state

Corrections of the chemical potential and the inverse compressibility can be calculated from Eqs. (19) and (20) in the same way as the condensate depletion. The analytic results for the correction of the chemical potential are given by

$$\Delta_\mu(\epsilon = 0) = -\frac{4(z_\rho^2 - 2)}{(z_\rho^2 - 1)(z_\rho + 1)(z_\rho + z_z)} T\left(\frac{z_\rho - 1}{z_\rho + 1} \frac{z_z - z_\rho}{z_z + z_\rho}\right) + \frac{8}{z_\rho + z_z} T\left(\frac{z_\rho - z_z}{z_\rho + z_z}\right), \quad (55)$$

$$\begin{aligned} \Delta_\mu(z_\rho = z_z = z) = & 2 \frac{-1 + \lambda + 2z\delta\{-1 - \lambda + z\delta[1 - \lambda + z\delta(1 + \lambda)]\}}{z(-1 + z^2\delta^2)[1 - \lambda + z\delta(1 + \lambda)]} \\ & + \frac{2(-1 + \lambda)}{z^2\delta} T(-\lambda) + \frac{2(-1 + \lambda)}{z^2\delta(-1 + z\delta)(1 + z\delta)^2} T\left(\frac{z\delta - 1}{z\delta + 1} \lambda\right). \end{aligned} \quad (56)$$

The disorder corrections of the chemical potential are shown in Fig. 2. The correction increases with increasing disorder, regardless of the strength of the dipolar interaction and disorder correlation lengths. This is due to the

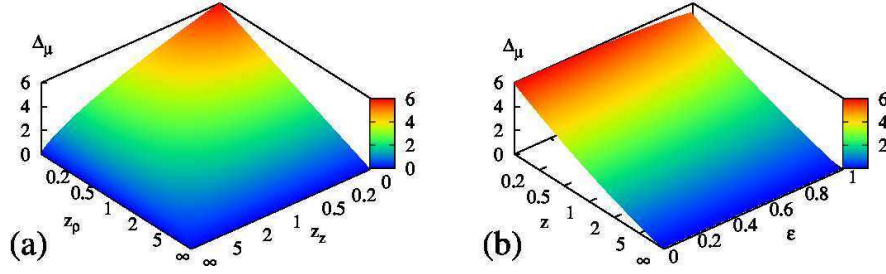


FIG. 2: (Color online) Change in the chemical potential due to weak disorder for: (a) contact interaction and anisotropic disorder from Eq. (55) and (b) isotropic disorder and dipolar interaction from Eq. (56).

repulsive interparticle interaction, which has a higher potential energy when the fluid is less uniform. The correction of the chemical potential in the case of the contact interaction, shown in Fig. 2 a), has a similar dependence on the correlation lengths as the condensate depletion in Fig. 1 a), while, according to Fig. 2 b), the dipole-dipole interaction does not have a significant effect. This is due to the fact that the dipolar interaction contributes partially as attractive and partially as repulsive, thus leading only to a small net effect. Note that the chemical potential in the clean case is anisotropic as can be seen from Eq. (19) and the directional dependence of the limit $\mathbf{k} \rightarrow \mathbf{0}$ in Eq. (45). This is discussed in more detail in Ref. [50].

The corresponding results for the inverse compressibility are:

$$\Delta_{\frac{\partial \mu}{\partial n}}(\epsilon = 0) = \frac{2}{(-1 + z_\rho^2)^2} \left[\frac{3 + z_z(2 + z_\rho^2)}{2(1 + z_z)^2} + \frac{z_\rho^2(-4 + z_\rho^2)}{(1 + z_\rho)(z_z + z_\rho)} T \left(\frac{z_\rho - 1}{z_\rho + 1} \frac{z_z - z_\rho}{z_z + z_\rho} \right) \right], \quad (57)$$

$$\Delta_{\frac{\partial \mu}{\partial n}}(z_\rho = z_z = z) = \frac{(-1 + \lambda) \{1 - \lambda + z^2 \delta^2 [2 - 2\lambda + 3z\delta(1 + \lambda)]\}}{z(-1 + z^2 \delta^2)^2 [1 - \lambda + z\delta(1 + \lambda)]^2}. \quad (58)$$

They represent intermediate results for calculating the sound velocity in subsection III D, according to Eq. (42).

C. Superfluidity

The disorder correction of the superfluid density tensor can be separated into a perpendicular and a parallel component after integration with respect to φ , using Eq. (32) we get

$$-\hat{\Delta}_{n_S} = -2\Delta_{n_0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2I_{sd} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad (59)$$

where Δ_{n_0} is already calculated in Eqs. (53) and (54), and I_{sd} is a new integral of the form

$$I_{sd} = \int_0^1 dt \frac{t^2}{v(1 + vt)^2}, \quad (60)$$

which yields in the two studied special cases the following solutions:

$$I_{sd}(\epsilon = 0) = \frac{1}{(-z_z + z_\rho)(z_z + z_\rho)} \left[-\frac{2 + z_z}{1 + z_z} + \frac{2(-2 + z_\rho^2)}{(1 + z_\rho)(z_z + z_\rho)} T \left(\frac{z_\rho - 1}{z_\rho + 1} \frac{z_z - z_\rho}{z_z + z_\rho} \right) + \frac{4}{z_z + z_\rho} T \left(\frac{z_\rho - z_z}{z_\rho + z_z} \right) \right], \quad (61)$$

$$I_{sd}(z_\rho = z_z = z) = \frac{(-1 + \lambda)^3}{4z\delta^2\lambda[1 - \lambda + z\delta(1 + \lambda)]} - \frac{(-1 + \lambda)^3}{4z^2\delta^3\lambda} T(-\lambda) + \frac{(-1 + \lambda)^3}{4z^2\delta^3(\lambda + z\delta\lambda)} T \left(\frac{z\delta - 1}{z\delta + 1} \lambda \right). \quad (62)$$

In an arbitrary direction of an unit vector \mathbf{q} the superfluid density can be calculated using the tensorial superfluid density by $n_S(\mathbf{q}) = \mathbf{q} \hat{n}_S \mathbf{q}$. In the case of the cylindrical symmetry it is reduced to

$$n_S(\mathbf{q}) = n_{S_\rho} \sin^2 \phi(\mathbf{q}, \mathbf{e}_z) + n_{S_z} \cos^2 \phi(\mathbf{q}, \mathbf{e}_z). \quad (63)$$

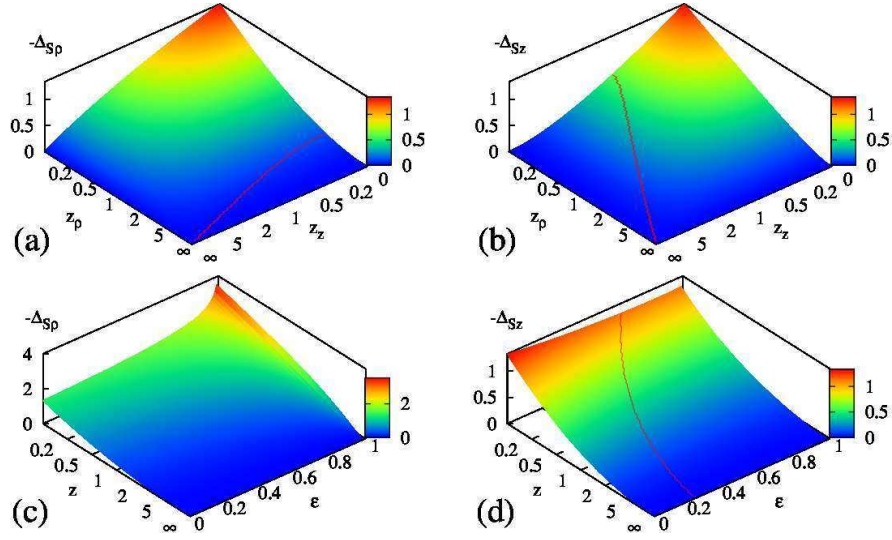


FIG. 3: (Color online) Change in the (a), (c) perpendicular and (b), (d) parallel superfluid density due to weak disorder for (a), (b) contact interaction and anisotropic disorder and (c), (d) dipolar interaction and isotropic disorder. Red lines show values of parameters where the superfluid depletion becomes equal to the condensate depletion.

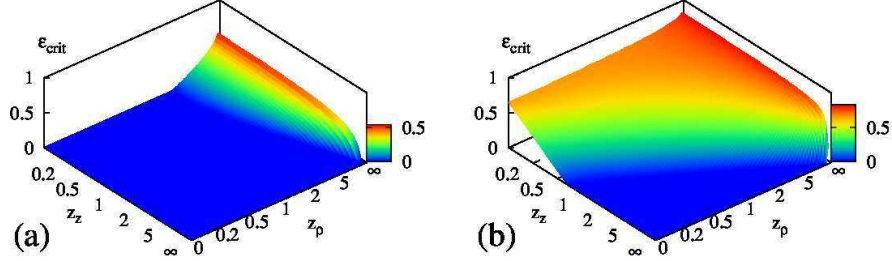


FIG. 4: (Color online) Critical values of the parameter ϵ for (a) perpendicular and (b) parallel superfluid density, at which the superfluid depletion becomes equal to the condensate depletion.

Thus, obtaining the disorder corrections for n_{S_ρ} and n_{S_z} is sufficient for recovering the superfluid depletion in any direction. From Eq. (59) we directly read off

$$-\Delta_{S_\rho} = -2\Delta_{n_0} - 2I_{sd}, \quad (64)$$

$$-\Delta_{S_z} = 2I_{sd}. \quad (65)$$

Note that the “-” sign in front of $\Delta_{S_{\rho,z}}$ and Δ_{n_0} suggest that the change of the superfluid densities and the condensate density are negative, or, equivalently that the depletion is positive. For isotropic systems Δ_{S_ρ} and Δ_{S_z} are both equal to $\frac{4}{3}\Delta_{n_0}$, as can be seen from Eq. (33). Due to an anisotropy, however, there is a range of correlation lengths and relative dipolar interaction strength, where the superfluid depletion is smaller than the condensate depletion. Some particles from the fragmented fluid contribute to superfluidity, suggesting that they are not localized indefinitely, but have some finite localization time, which was calculated in Ref. [32] within the Hartree-Fock theory of dirty bosons. The superfluid depletion in the case of contact interaction and anisotropic disorder shows a similar behaviour as the condensate depletion, as can be seen in Figs. 3 a) and 3 b). For isotropic disorder and dipolar interaction the depletion of the perpendicular component is similar to the condensate depletion Fig. 3 c), but the depletion of the parallel component decreases as the relative interaction strength increases, as shown in Fig. 3 d). The red lines in Fig. 3 show where the superfluid depletion becomes equal to the condensate depletion. Fig. 4 shows the values of ϵ for which this change happens.

Although defined only for systems without a trap, the above calculated superfluid density could be extended to the trapped case in the simplest way by assuming that it depends only on the local density. If we turn on a slowly moving disorder for a short time τ , such that $\mathbf{v}\tau$ is much smaller than the size of the trap, before switching off the trap, this would change the momentum distribution which could be afterwards reconstructed from a time-of-flight measurement. In this way our predictions for the superfluid density in such a system might become observable in

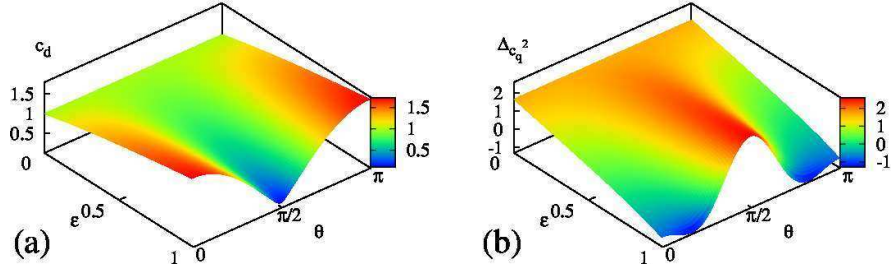


FIG. 5: (Color online) Sound velocity for dipolar interaction (a) clean case and (b) correction due to weak delta-correlated disorder.

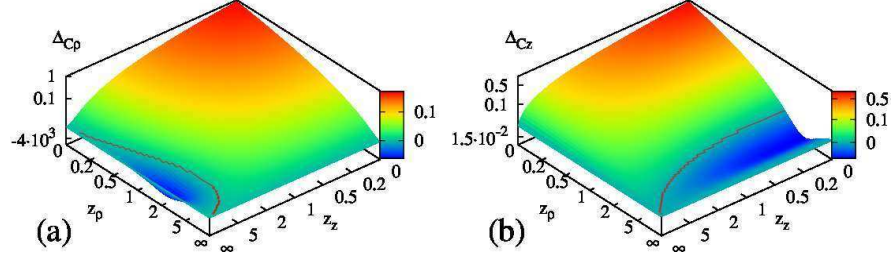


FIG. 6: (Color online) Corrections of (a) perpendicular and (b) parallel sound velocity for weak anisotropic disorder and contact interaction. The red lines show values of the correlation lengths and the relative dipole-dipole interaction for which the correction vanishes.

experiment.

D. Sound velocity

The corresponding disorder correction of the sound velocity in the direction of the unit vector \mathbf{q} can be calculated using Eqs. (42) and (46):

$$c_{\mathbf{q}}^2 = \frac{gn}{m} \left[\frac{V_{\mathbf{q}}(\mathbf{0})}{g} + \frac{n_{\text{HM}}}{n} \left(-\frac{V_{\mathbf{q}}(\mathbf{0})}{g} \mathbf{q}^T \hat{\Delta}_{n_N} \mathbf{q} + \Delta_{\frac{\partial \mu}{\partial n}} \right) + \dots \right], \quad (66)$$

which leads to

$$\Delta c_{\mathbf{q}}^2 = -\frac{V_{\mathbf{q}}(\mathbf{0})}{g} \mathbf{q}^T \hat{\Delta}_{n_N} \mathbf{q} + \Delta_{\frac{\partial \mu}{\partial n}}, \quad (67)$$

where $V_{\mathbf{q}}(\mathbf{0}) = \lim_{k \rightarrow 0} V(k\mathbf{q})$ denotes the directional dependence of the potential V on \mathbf{q} , according to Eq. (45). The anisotropy factor due to the dipolar interaction $c_d = V_{\mathbf{q}}(\mathbf{0})/g$ and the delta-correlated disorder correction [53] are plotted in Fig. 5.

The anisotropy of disorder comes into play in a simple way. From Eqs. (42) and (63) it follows that the sound velocity can also be separated into a parallel and a perpendicular component

$$c^2(\mathbf{q}) = c_{\rho}^2 \sin^2 \phi(\mathbf{q}, \mathbf{e}_z) + c_z^2 \cos^2 \phi(\mathbf{q}, \mathbf{e}_z), \quad (68)$$

with

$$c_{\rho,z}^2 = \frac{1}{m} \frac{\partial \mu}{\partial n} n_{S_{\rho,z}} \quad (69)$$

and the corresponding dimensionless disorder correction

$$\Delta c_{\rho,z}^2 = \Delta_{\frac{\partial \mu}{\partial n}} + \Delta_{S_{\rho,z}}. \quad (70)$$

By introducing a weak disorder, the sound velocity changes via two competing effects: the decrease of the compressibility, i.e. the increase in the inverse compressibility, from Eq. (20), which tends to increase the sound velocity, and

the decrease of the superfluid density (negative value of $\Delta_{S_{\rho,z}}$), which tends to decrease the sound velocity. The results are shown in Fig. 6. The red line shows where the second order sound velocity correction vanishes. For small disorder correlation lengths the decrease in compressibility is dominant. These corrections can be experimentally measured by determining the phonon dispersion relation by using Bragg spectroscopy [2, 60, 61].

IV. CONCLUSIONS

We have analyzed in detail how the anisotropy of both the dipolar interaction and the presence of disorder affects the directional dependence of different physical observables of dirty Bose-Einstein condensates. Using at zero temperature the mean-field approach, we have calculated the condensate depletion due to disorder, as well as the corresponding corrections to the equation of state, the superfluid density, and the sound velocity. In particular, we have discussed the consequences for the superfluid density, which becomes a tensorial quantity as a linear response to the moving disorder. Whereas Ref. [53] analyzed a dipolar BEC in isotropic disorder potential, we have shown here that the anisotropic disorder provides a separate origin for the tensorial nature of the superfluid density. We have found that a large enough disorder anisotropy can even make both the parallel and perpendicular superfluid density component larger than the corresponding condensate density, which happens in the case of dipolar interaction and isotropic disorder only for the parallel component [53]. These initial results necessitate further studies, as they contribute to the overall physical picture in which the localization of bosons in the respective minima of the disorder potential occurs at a characteristic time scale [32]. This localization time remains to be analyzed in more detail in a forthcoming publication.

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